

$$\int_x^{hx} g^* \alpha - \alpha$$

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ABSTRACT. Let  $X$  be a path-connected topological space admitting a universal cover. Let  $\alpha$  be a degree one cohomology class on  $X$ . We define and study a two-cocycle on a group acting on  $X$  by homeomorphisms preserving the class  $\alpha$ .

We apply the cocycle to the study of the distortion in the group of homeomorphisms preserving the class  $\alpha$ . In particular, we introduce a local rotation number of a homeomorphism and prove that a homeomorphism with non-constant local rotation number is undistorted. We also use the cocycle to investigate group actions on  $X$ . For example, we show that if an action preserves a Borel probability measure on  $X$  then the cocycle is cohomologically trivial.

## 1. INTRODUCTION AND THE STATEMENT OF THE RESULTS

{S:intro}

Let  $X$  be a path-connected topological space admitting a universal cover. Let  $\text{Homeo}(X, \alpha)$  denote the group of homeomorphisms of  $X$  preserving a cohomology class  $\alpha \in H^1(X; \mathbf{A})$ , where  $\mathbf{A}$  is a trivial coefficient system. In the present paper, we study distortion in  $\text{Homeo}(X, \alpha)$  and properties of group actions on  $X$  by homeomorphisms preserving the class  $\alpha$ .

The main tool is a two-cocycle on the group  $\text{Homeo}(X, \alpha)$  defined by the following formula

$$\mathfrak{G}_{x, \alpha}(g, h) := \int_{\gamma} g^* \alpha - \alpha,$$

where  $x \in X$  is a reference point,  $\gamma$  is a path from  $x$  to  $hx$ , and  $\alpha$  is a singular cocycle representing the class  $\alpha \in H^1(X, \mathbf{A})$ . The expression  $\int_{\gamma} \sigma$  denotes the natural pairing between a chain  $\gamma$  and a cochain  $\sigma$ . We shall frequently omit one or both subscripts when it does not lead to a confusion.

The results obtained in the paper can be divided into two parts. The first (Section 3) is about distortion. In the second part (Sections 4–6) we study the cohomology class of  $\mathfrak{G}$  and properties of group actions on  $X$  depending on either vanishing or non-vanishing of the cohomology class of  $\mathfrak{G}$ .

Throughout the paper we consider groups of homeomorphisms equipped either with compact-open or discrete topology. The latter case is marked with the superscript  $\delta$ . By  $G_0$  we denote the connected component of the identity of  $G$ . Let us now discuss the main results.

**1.A. Distortion in groups.** Let  $\Gamma$  be a finitely generated group. Define the word norm associated with fixed set of generators  $S$  to be

$$|g| := \min\{k \in \mathbb{N} \mid g = s_1 \dots s_k, s_i \in S\}.$$

The **translation length** of an element  $g \in \Gamma$  is defined to be

$$\tau(g) := \lim_{n \rightarrow \infty} \frac{|g^n|}{n}.$$

(Note that by the subadditivity property of the length, the limit exists.) An element  $g \in \Gamma$  is called **undistorted** if its translation length is positive and this property does not depend on the choice of generators. If  $G$  is a general (not necessarily finitely generated) group then  $g \in G$  is called **undistorted** if it is undistorted in every finitely generated subgroup of  $G$ . Notice that distortion in a subgroup implies distortion in an ambient group.

It is well known that certain lattices in semisimple Lie groups contain distorted elements due to a result of Lubotzky, Mozes, and Raghunathan [?]. On the other hand, the distortion in groups of diffeomorphisms of closed manifolds is rare as shown, for example, by Franks and Handel [?], Gambaudo and Ghys [?], or Polterovich [?]. This provides restrictions on possible actions of such lattices.

The papers cited above are concerned with the distortion either in volume preserving or in Hamiltonian diffeomorphisms. It follows from our results, however, that many elements are undistorted in groups of homeomorphisms of manifolds of dimension at least two and with nontrivial first real cohomology. Essentially, this is as much as one gets for such manifolds. In contrast, Calegari and Freedman [?, Theorem C] proved that all homeomorphisms of the sphere  $\mathbb{S}^n$  are distorted in  $\text{Homeo}(\mathbb{S}^n)$ .

The results presented in this section are consequences of a more general Theorem 3.4. The proof of the following theorem is presented in Section 3.B.

{T:distortion}

**Theorem 1.1.** *Let  $X$  be compact and let  $g \in G \subseteq \text{Homeo}(X, \alpha)$  where  $\alpha \in H^1(X; \mathbb{R})$  is represented by a one-cocycle  $\alpha$ . Suppose that  $g$  has two fixed points  $x, y \in X$  and let  $\gamma$  be a path from  $x$  to  $y$ . If*

$$\int_{\gamma} g^* \alpha - \alpha \neq 0$$

*then  $g$  is undistorted in  $G$ .*

Two fixed points  $x, y$  of a map  $g: X \rightarrow X$  are called **Nielsen equivalent** if there exists a path  $\gamma$  from  $x$  to  $y$  such that  $\gamma$  and  $g\gamma$  are homotopic modulo the endpoints. The hypothesis of the above theorem implies that the homeomorphism  $g$  has two fixed points which are Nielsen nonequivalent in a stronger sense. Namely, the cycle  $g\gamma - \gamma$  is homologically nontrivial.

{E:surface\_niels}

**Example 1.2.** Let  $G \subset \text{Homeo}(\Sigma)$  be a group of homeomorphisms of a closed oriented surface  $\Sigma$  acting trivially on the first cohomology of  $\Sigma$ . Suppose that  $g \in G$  has two fixed points  $x, y \in \Sigma$  such that  $g\gamma - \gamma$  is a homologically nontrivial loop, where  $\gamma$  is a path from  $x$  to  $y$ . Then  $g$  is undistorted in  $G$ . Indeed, there exists a cohomology class  $\alpha \in H^1(\Sigma; \mathbb{Z})$  evaluating nontrivially on  $g\gamma - \gamma$ .  $\diamond$

The cocycle  $\mathfrak{G}$  depends on a reference point and it is unbounded in general. In Section 3.C we define a local rotation number of a homeomorphism with respect to a point at which the cocycle  $\mathfrak{G}$  is bounded.

{T:rotation}

**Theorem 1.3.** Let  $\alpha \in H^1(X; \mathbb{Z})$  and let  $g \in \text{Homeo}(X, \alpha)$  and assume that  $X$  is compact. Let  $x$  and  $y$  be points such that the cocycles  $\mathfrak{G}_x$  and  $\mathfrak{G}_y$  are bounded on the cyclic subgroup generated by  $g$ . If the local rotation numbers of  $g$  at  $x$  and  $y$  are distinct then  $g$  is undistorted in  $\text{Homeo}(X, \alpha)$ .

{E:annulus}

**Example 1.4.** Let  $X = \mathbb{S}^1 \times [0, 1]$  be the closed annulus and let  $h: X \rightarrow X$  be a homeomorphism preserving the orientation and the components of the boundary. If the topological rotation numbers of  $h$  restricted to the boundary circles are distinct then  $h$  is undistorted in the group of orientation preserving homeomorphisms of  $X$ .  $\diamond$

We discuss more applications of Theorem 1.3 in Section 3.E.

**1.B. Vanishing properties and dynamics.** Let us discuss the conditions under which the cohomology class of  $\mathfrak{G}$  is trivial. This has a dynamical flavour as nonvanishing of  $[\mathfrak{G}]$  is an obstruction to the existence of certain invariant objects. The proof of the next result is presented in Section 4.A.

{T:subset}

**Theorem 1.5.** Let  $i_L: L \rightarrow X$  be the inclusion of a path-connected subset such that  $i_L^* \alpha = 0 \in H^1(L; \mathbb{A})$ . If an action  $\psi: G \rightarrow \text{Homeo}(X, \alpha)$  preserves the subset  $L$  then  $\psi^*[\mathfrak{G}] = 0$ .

One can think of this result as a form of ergodicity in which invariant subsets have necessarily complicated topology. In particular, if  $\psi^*[\mathfrak{G}]$  is nontrivial then the action preserves no path-connected and simply-connected subsets.

{T:measure}

**Theorem 1.6.** *Suppose that  $X$  is compact. If the action  $\psi: G \rightarrow \text{Homeo}(X, \alpha)$  preserves a Borel probability measure on  $X$  then the class  $\psi^*[\mathfrak{G}] \in H^2(BG^\delta; \mathbf{R})$  is trivial.*

The proof is presented in Section 4.B.

A topological group  $G$  is called **amenable** if a continuous affine action of  $G$  on a non-empty compact convex subset of a locally convex topological vector space has a fixed point [?, Theorem G.1.7].

{C:amenable}

The space of Borel probability measures on a compact space is compact as a subset of the dual of the Banach space of continuous functions on  $X$  equipped with weak-\* topology. Consequently, every action of an amenable group on a compact space preserves a Borel probability measure. This proves the following result.

**Corollary 1.7.** *Let  $X$  be compact. If  $\psi: G \rightarrow \text{Homeo}(X, \alpha)$  is an action of a topological amenable group then  $\psi^*[\mathfrak{G}] = 0$  in  $H^2(BG^\delta; \mathbf{R})$ .*  $\square$

**Example 1.8.** Let  $X = U(1)$  and consider the natural action of  $U(1)$  on itself. Then  $[\mathfrak{G}]$  is nontrivial in  $H^2(BU(1)^\delta; \mathbf{Z})$  according to Theorem 5.1 and trivial in  $H^2(BU(1)^\delta; \mathbf{R})$  by the above corollary.  $\diamond$

1.C. **Nonvanishing properties.** Let  $\text{ev}: G \rightarrow X$  denote the evaluation at the reference point  $x \in X$  associated with an action of a topological group  $G$  on  $X$ . Our main result about the nonvanishing of the class  $[\mathfrak{G}]$  is concerned with the homology of the evaluation map (see Section 6.A for the proof).

{T:ev}

**Theorem 1.9.** *Let  $\psi: G \rightarrow \text{Homeo}(X, \alpha)$  be an action of a connected topological group. Assume that the homomorphism  $H^2(BG; \mathbf{A}) \rightarrow H^2(BG^\delta; \mathbf{A})$  induced by the identity on  $G$  is injective. Then the class  $\psi^*[\mathfrak{G}] \in H^2(BG^\delta; \mathbf{A})$  is nonzero if and only if  $\text{ev}^*(\alpha) \in H^1(G; \mathbf{A})$  is nontrivial.*

There are cases in which the hypothesis of Theorem 1.9 is satisfied. For example, let  $X$  be a connected topological manifold with trivial ends. It is a result of McDuff [?] that the identity homomorphism induces an isomorphism  $H^*(BG; \mathbf{A}) \rightarrow H^*(BG^\delta; \mathbf{A})$ , where  $G$  is a group of homeomorphisms of  $X$  containing the component of the identity.

{C:homeo}

**Corollary 1.10.** *Let  $X$  be a connected topological manifold with trivial ends. Consider the natural action of  $G = \text{Homeo}(X)_0$ , the connected component of the identity of the group of homeomorphisms of  $X$ . Then the class  $[\mathfrak{G}] \in H^2(BG^\delta; \mathbf{A})$  is nonzero if and only if  $\text{ev}^*(\alpha) \neq 0$ .*  $\square$

Another instance where the hypothesis of Theorem 1.9 is satisfied is when  $G$  is a connected perfect group with countable fundamental group. The following result is proven in Section 6.C.

$\{C: \text{perfect}\}$

**Corollary 1.11.** *Let  $\psi: G \rightarrow \text{Homeo}(X, a)$  be an action of a connected perfect group with countable fundamental group. Then  $\psi^*[\mathfrak{G}] \neq 0$  if and only if  $\text{ev}^*(a) \neq 0$ .*

Let  $K$  be a connected compact Lie group. Then the evaluation map of the action of  $K$  on itself induces an isomorphism  $\text{ev}^*: H^1(K; \mathbf{A}) \rightarrow H^1(K; \mathbf{A})$  for any coefficients  $\mathbf{A}$ . If there exist a nontrivial homomorphism  $\pi_1(K) \rightarrow \mathbf{A}$  then  $H^1(K; \mathbf{A}) = \text{Hom}(\pi_1(K), \mathbf{A})$  is nontrivial. Let  $G$  be a connected algebraic semisimple Lie group and let  $G = KAN$  be its Iwasawa decomposition. Then  $G$  acts on  $K = G/AN$  and the action extends the natural action of  $K$  on itself. Since  $G$  is homotopy equivalent to  $K$  we obtain that the action induces an isomorphism  $\text{ev}^*: H^1(G; \mathbf{A}) \rightarrow H^1(K; \mathbf{A})$ . Applying this observation and Corollary 1.11 gives the following application.

**Corollary 1.12.** *Let  $\psi: G \rightarrow \text{Homeo}_0(K)$  be the action of a connected algebraic semisimple Lie group on its maximal compact subgroup  $K \subset G$ . Then the class  $\psi^*[\mathfrak{G}_\alpha] \in H^2(BG^\delta; \mathbf{A})$  associated with the action is nontrivial for any nonzero element  $[\alpha] \in H^1(K; \mathbf{A})$ .  $\square$*

**Example 1.13.** Let  $\Gamma \subset G$  be a torsion free uniform lattice in a connected non-compact simple Lie group of Hermitian type. Then  $B\Gamma = \Gamma \backslash G/K$  is a closed Kähler manifold and the homomorphism  $H^2(BG^\delta; \mathbf{R}) \rightarrow H^2(B\Gamma; \mathbf{R})$  is injective. In this case the class of the cocycle  $\mathfrak{G}$  is nontrivial in  $H^2(B\Gamma; \mathbf{R})$  with respect to the action of  $\Gamma$  on  $K$  (as in the above corollary). The evaluation homomorphism is necessarily trivial since  $\Gamma$  is discrete.  $\diamond$

$\{E: \text{maps}\}$

**Example 1.14.** Let  $X = \text{Map}_1(S^1, S^1)$  be the space of continuous degree one self-maps of the circle. The group  $G = \text{Homeo}(S^1)_0$  acts on  $X$  by the reparametrisations. Let  $a \in H^1(X; \mathbf{Z})$  be the class equal to the pull back of a generator of  $H^1(S^1; \mathbf{Z})$  with respect to the evaluation at a point. Then  $\text{ev}^*(a) \neq 0$  because the composition  $SO(2) \subset G \xrightarrow{\text{ev}} X \rightarrow S^1$  is equal to the identity. Since, moreover, the group  $G$  is perfect, Corollary 1.11 applies and the corresponding cocycle  $\mathfrak{G}$  is cohomologically nontrivial on  $G$ .  $\diamond$

Let us finish this section with an example of a tautological construction providing actions with nontrivial class of the cocycle  $\mathfrak{G}$ .

**Example 1.15.** Consider a central extension of discrete groups

$$0 \rightarrow \mathbf{A} \rightarrow \widehat{G} \rightarrow G \rightarrow 1$$

and observe that  $G$  acts on the space  $X := \mathbf{A} \backslash E\widehat{G}$ . It follows from Theorem 5.1 that the cohomology class of the cocycle  $\mathfrak{G}$  associated with this action is equal to the class  $\mathfrak{C} \in H^2(BG^\delta; \mathbf{A})$  of the above central

extension. This shows that every class in the second cohomology of a discrete group is equal to the class of the cocycle  $\mathfrak{G}$  associated with an appropriate action.  $\diamond$

**Historical remarks.** The cocycle  $\mathfrak{G}$  can be defined for an arbitrary, not necessarily closed, one-cochain  $\alpha$  on a suitably defined subgroup of  $\text{Homeo}(X)$ . It has been first defined by Ismagilov, Losik, and Michor [?] for a primitive of a symplectic form and further studied by the authors in [?].

The cocycle  $\mathfrak{K}_\alpha$  (see Section 2.C for definition) appears in Gambaudo and Ghys [?] and in Arnold and Khesin [?, p. 247] in the case of a symplectic ball. It has been studied for a general symplectically aspherical manifold in [?].

The local rotation number generalizes the rotation number of a homeomorphism of a circle. There are related notions in the literature. For example the rotation defined by Burger, Iozzi, and Wienhard in [?, Definition 7.1] or the rotation vector of Franks [?, Definition 2.1].

## 2. COCYCLES ON $\text{Homeo}(X, \alpha)$

{S:cocycles}

In this section we discuss the basic properties of the cocycle  $\mathfrak{G}$ . We also define an auxiliary one-cocycle  $\mathfrak{K}_\alpha$  with values in the space of functions on  $X$  modulo constants. The main result of this section is Lemma 2.4 stating that under suitable assumptions  $\mathfrak{K}_\alpha$  takes values in the space of *continuous* functions on  $X$  modulo constants.

Let  $X$  be a path-connected, topological space admitting a universal cover  $\tilde{X} \rightarrow X$ . Let  $\alpha \in H^1(X; \mathbf{A})$  be a cohomology class, where  $\mathbf{A}$  is an Abelian group of trivial coefficients. Let  $G \subseteq \text{Homeo}(X, \alpha)$  be a group of homeomorphisms of  $X$  preserving the class  $\alpha$ .

{SS:explicit}

**2.A. An explicit formula for the cocycle  $\mathfrak{G}$ .** Let  $\alpha \in Z^1(X; \mathbf{A})$  be a singular one-cocycle representing the class  $\alpha$ . Let  $x \in X$  be a reference point. Define  $\mathfrak{G}_{x,\alpha}: G \times G \rightarrow \mathbf{A}$  by

$$\mathfrak{G}_{x,\alpha}(g, h) := \int_\gamma g^* \alpha - \alpha$$

where  $\gamma$  is a path from  $x$  to  $hx$ . Recall that the expression  $\int_\gamma \sigma$  denotes the natural pairing of a chain  $\gamma$  and a cochain  $\sigma$ . We find this nonstandard notation useful because in many concrete examples presented in the paper the cocycle  $\alpha$  is defined by the integration of a differential form over smooth paths.

{L:G} The following basic properties are straightforward to prove.

**Lemma 2.1.**

- (1) The function  $\mathfrak{G}_{x,\alpha}$  is a two-cocycle on  $\text{Homeo}(X, \mathfrak{a})$ . That is it satisfies the following identity:

$$\mathfrak{G}_{x,\alpha}(h, k) - \mathfrak{G}_{x,\alpha}(gh, k) + \mathfrak{G}_{x,\alpha}(g, hk) - \mathfrak{G}_{x,\alpha}(g, h) = 0.$$

- (2) The value  $\mathfrak{G}_{x,\alpha}(g, h)$  does not depend on the choice of a path from  $x$  to  $hx$ .

- (3) The cohomology class of the cocycle  $\mathfrak{G}_{x,\alpha}$  depends neither on the choice of the reference point  $x$  nor on the choice of the cocycle  $\alpha$ .

- (4) If either  $g$  preserves  $\alpha$  or  $h$  preserves  $x$  then  $\mathfrak{G}_{x,\alpha}(g, h) = 0$ .  $\square$

{SS:singular}

**2.B. On singular one-cocycles.** The results of this section are used to prove Lemma 2.4.

Since  $H^1(X; \mathbf{A}) = \text{Hom}(\pi_1(X), \mathbf{A})$  one can define a cover

$$\mathbf{A} \rightarrow X_{\mathfrak{a}} := \tilde{X} \times_{\pi_1(x)} \mathbf{A} \rightarrow X,$$

where  $\tilde{X}$  is the universal cover of  $X$  and  $\pi_1(x)$  acts on  $\mathbf{A}$  via homomorphism defined by  $\mathfrak{a}$ . In what follows, the action  $\mathbf{A} \times X_{\mathfrak{a}} \rightarrow X_{\mathfrak{a}}$  by the deck transformations will be denoted additively:  $(a, z) \mapsto a + z$ .

Let  $x \in X$  be a reference point in  $X$  and let  $\tilde{x} \in p^{-1}(x)$  be a reference point in  $X_{\mathfrak{a}}$ . Let  $\alpha$  be a singular cocycle representing the class  $\mathfrak{a}$ . That is,  $\alpha$  is a homomorphism  $C_1(X; \mathbf{A}) \rightarrow \mathbf{A}$  defined on the group of chains on  $X$  with the coefficients in  $\mathbf{A}$ . It defines an  $\mathbf{A}$ -equivariant map  $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$  in the following way. Given a point  $\tilde{y} \in p^{-1}(y)$  let  $\gamma: [0, 1] \rightarrow X$  be a path from  $x$  to  $y$ . Let  $\tilde{\gamma}: [0, 1] \rightarrow X_{\mathfrak{a}}$  be its lift such that  $\tilde{\gamma}(0) = \tilde{x}$ . Then we define  $\mathbf{a}(\tilde{y})$  as the unique element such that  $\int_{\gamma} \alpha + \tilde{y} = \mathbf{a}(\tilde{y}) + \tilde{\gamma}(1)$ . If we put  $\tilde{y} := \tilde{\gamma}(1)$  we obtain that

$$\mathbf{a}(\tilde{\gamma}(1)) = \int_{\gamma} \alpha.$$

Let us check that  $\mathbf{a}$  does not depend on the choice of the path  $\gamma$ . Let  $\gamma_{\pm}$  be two paths from  $x$  to  $y$  and let  $\mathbf{a}_{-}$  and  $\mathbf{a}_{+}$  denote the corresponding maps. By letting  $\tilde{y} = \tilde{\gamma}_{+}(1)$  in the equality

$$\int_{\gamma_{+}} \alpha + \tilde{\gamma}_{-}(1) = \int_{\gamma_{-}} \alpha + \tilde{\gamma}_{+}(1)$$

we get

$$\int_{\gamma_{+}} \alpha + \tilde{\gamma}_{-}(1) = \int_{\gamma_{-}} \alpha + \tilde{y}$$

which shows that  $\mathbf{a}_{+}(\tilde{y}) = \int_{\gamma_{+}} \alpha = \mathbf{a}_{-}(\tilde{y})$  as claimed.

The equivariance of  $\mathbf{a}$  is immediate from the definition. Another choice of a reference point results in changing  $\mathbf{a}$  by an additive constant.



Let  $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$  be an  $\mathbf{A}$ -equivariant function. Let  $\gamma: [0, 1] \rightarrow X$  be a path and let  $\tilde{\gamma}: [0, 1] \rightarrow X_{\mathfrak{a}}$  be its lift. The following formula defines a singular one-cocycle with values in  $\mathbf{A}$ .

{L:singular}

$$\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1)) - \mathbf{a}(\tilde{\gamma}(0))$$

**Lemma 2.2.** *The above constructions are inverse to each other and hence provide a bijective correspondence between singular one-cocycles in the class  $\mathfrak{a} \in H^1(X, \mathbf{A})$  and  $\mathbf{A}$ -equivariant maps  $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$  up to the constants.*

*Proof.* Let  $\alpha$  be a singular one-cocycle representing the class  $\mathfrak{a}$ . It defines an equivariant map  $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$  such that  $\int_{\gamma} \alpha + \tilde{\gamma} = \mathbf{a}(\tilde{\gamma}) + \tilde{\gamma}(1)$  for every path  $\gamma: [0, 1] \rightarrow X$  from  $x$  to  $y$ . We need to check that  $\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1)) - \mathbf{a}(\tilde{\gamma}(0))$ .

Let  $\tilde{y} := \tilde{\gamma}(1)$  where the lift  $\tilde{\gamma}$  is chosen so that  $\mathbf{a}(\tilde{\gamma}(0)) = 0$ . Then

$$\int_{\gamma} \alpha + \tilde{\gamma}(1) = \mathbf{a}(\tilde{\gamma}(1)) + \tilde{\gamma}(1)$$

implies that  $\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1))$ .

Conversely, let  $\mathbf{a}: X_{\mathfrak{a}} \rightarrow \mathbf{A}$  be an  $\mathbf{A}$ -equivariant map. It defines a singular cocycle  $\alpha$  by the identity  $\int_{\gamma} \alpha = \mathbf{a}(\tilde{\gamma}(1))$ , where  $\tilde{\gamma}$  is a lift of  $\gamma$  such that  $\mathbf{a}(\tilde{\gamma}(0)) = 0$ . We then clearly get that  $\int_{\gamma} \alpha + \tilde{\gamma}(1) = \mathbf{a}(\tilde{\gamma}(1)) + \tilde{\gamma}(1)$ .  $\square$

**2.C. The one-cocycle  $\mathfrak{K}_{\alpha}$ .** If  $f \in G \subset \text{Homeo}(X, \mathfrak{a})$  then  $f^* \alpha - \alpha$  is an exact singular one-cocycle on  $X$  and the identity  $\delta(\mathfrak{K}_{\alpha}(f)) = f^* \alpha - \alpha$  defines a map

$$\mathfrak{K}_{\alpha}: G \rightarrow C^0(X; \mathbf{A})/\mathbf{A}.$$

It is straightforward to check that  $\mathfrak{K}_{\alpha}$  is a one-cocycle (cf. [?, Proposition 2.3]). That is, it satisfies

$$\mathfrak{K}_{\alpha}(fg) = \mathfrak{K}_{\alpha}(f) \circ g + \mathfrak{K}_{\alpha}(g)$$

{L:G-K} for all  $f, g \in G$ .

**Lemma 2.3.** *If  $h$  and  $g$  are homeomorphisms preserving  $\mathfrak{a} = [\alpha]$  then*

$$\mathfrak{G}_{x, \alpha}(g, h) = \mathfrak{K}_{\alpha}(g)(hx) - \mathfrak{K}_{\alpha}(g)(x).$$

$\square$

*Proof.* It is an immediate consequence the definition of the cocycle  $\mathfrak{K}_{\alpha}$ . Indeed, we have

$$\mathfrak{G}_{x, \alpha}(g, h) = \int_x^{hx} g^* \alpha - \alpha = \int_x^{hx} \delta(\mathfrak{K}_{\alpha}(g)) = \mathfrak{K}_{\alpha}(g)(hx) - \mathfrak{K}_{\alpha}(g)(x).$$

$\square$



{L:continuous}

**Lemma 2.4.** Assume that  $X$  is paracompact. Let  $\mathfrak{a} \in H^1(X; \mathbf{R})$ . There exists a singular cocycle  $\alpha$  representing the class  $\mathfrak{a}$  such that for any homeomorphism  $h \in \text{Homeo}(X, \mathfrak{a})$  the function  $\mathfrak{K}_\alpha(h)$  is a continuous function.

*Remark 2.5.* If  $X$  is a differentiable manifold and  $\mathbf{A} = \mathbf{R}$  then every cohomology class is represented by a smooth and closed differential form  $\alpha$ . It follows that for any diffeomorphism  $h \in \text{Diff}(X, \mathfrak{a})$  the function  $\mathfrak{K}_\alpha(h)$  is smooth.

*Proof of Lemma 2.4.* Let us consider the real numbers  $\mathbf{R}$  endowed with the usual order topology and consider the bundle

$$\mathbf{R} \rightarrow E = \tilde{X} \times_{\pi_1 X} \mathbf{R} \xrightarrow{p} X.$$

Since the fibre is contractible and the base is paracompact it admits a continuous section  $s: X \rightarrow E$ . Such a section defines a continuous equivariant function  $\mathbf{a}: E \rightarrow \mathbf{R}$  by the identity  $p(\tilde{x}) = \mathbf{a}(\tilde{x}) + sp(\tilde{x})$ . Notice that  $X_\mathfrak{a} = \tilde{X} \times_{\pi_1 X} \mathbf{R}^\delta$  is the same set as  $E$  but with a finer topology. Thus  $\mathbf{a}: X_\mathfrak{a} \rightarrow \mathbf{R}$  is still a continuous function.

Let  $\tilde{g} \in \text{Homeo}(X_\mathfrak{a})$  be an  $\mathbf{R}$ -equivariant lift of  $g \in \text{Homeo}(X, \mathfrak{a})$ . Define a continuous function  $\hat{\mathfrak{K}}(g): X_\mathfrak{a} \rightarrow \mathbf{R}$  by

$$\hat{\mathfrak{K}}(g)(\tilde{x}) := \mathbf{a}(\tilde{g}\tilde{x}) - \mathbf{a}(\tilde{x}).$$

Since both  $\tilde{g}$  and  $\mathbf{a}$  are  $\mathbf{R}$ -equivariant the function  $\hat{\mathfrak{K}}(g)$  is  $\mathbf{R}$ -invariant and thus descends to a continuous function  $\mathfrak{K}(g): X \rightarrow \mathbf{R}$ .

Let us show that  $\mathfrak{K} = \mathfrak{K}_\alpha$ . Let  $\gamma$  be a path between  $x$  and  $y$ . Let  $\tilde{\gamma}$  be its lift with endpoints at  $\tilde{x}$  and  $\tilde{y}$ . Then

$$\begin{aligned} \mathfrak{K}(g)(y) - \mathfrak{K}(g)(x) &= (\mathbf{a}(\tilde{g}\tilde{y}) - \mathbf{a}(\tilde{g}\tilde{x})) - (\mathbf{a}(\tilde{y}) - \mathbf{a}(\tilde{x})) \\ &= \int_{g\gamma} \alpha - \int_\gamma \alpha = \int_\gamma g^* \alpha - \alpha. \end{aligned}$$

□

### 3. DISTORTION IN GROUPS

{SSd distortion}

**3.A. Quasimorphisms.** Let  $q: G \rightarrow \mathbf{R}$  be a map defined on a group  $G$ . The **defect**  $D(q)$  of the map  $q$  is defined to be

$$D(q) := \sup_{g, h \in G} |q(g) - q(gh) + q(h)|.$$

If the defect of  $q$  is finite then  $q$  is called a **quasimorphism**. A quasimorphism  $q$  is called **homogeneous** if  $q(g^n) = nq(g)$  for all  $n \in \mathbf{Z}$  and

$g \in G$ . For every quasimorphism  $q$  the formula

$$\hat{q}(g) := \lim_{n \rightarrow \infty} \frac{q(g^n)}{n}$$

defines a homogeneous quasimorphism called the homogenisation of  $q$ . Moreover,  $|\hat{q}(g) - q(g)| \leq D$  for all  $g \in G$  [?, Lemma 2.21]. Thus  $q$  is unbounded if and only if so is its homogenisation.

**Proposition 3.1.** *Let  $\alpha \in H^1(X; \mathbf{R})$ . Let  $G \subseteq \text{Homeo}(X, \alpha)$  be a subgroup on which the cocycles  $\mathfrak{G}_x$  and  $\mathfrak{G}_y$  are bounded, for some  $x, y \in X$ . Then the map  $q: G \rightarrow \mathbf{R}$  defined by*

$$q(g) := \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x)$$

is a quasimorphism on  $G$  and  $D(q) \leq \|\mathfrak{G}_x - \mathfrak{G}_y\| \leq \|\mathfrak{G}_x\| + \|\mathfrak{G}_y\|$ , where  $\|\cdot\|$  denotes the supremum norm of a bounded function.

*Proof.* This is a straightforward computation using the cocycle identity for  $\mathfrak{K}_\alpha$ .

$$\begin{aligned} q(f) - q(fg) + q(g) &= \mathfrak{K}_\alpha(f)(y) - \mathfrak{K}_\alpha(f)(x) \\ &\quad - (\mathfrak{K}_\alpha(f)(gy) + \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(f)(gx) - \mathfrak{K}_\alpha(g)(x)) \\ &\quad + \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) \\ &= \mathfrak{K}_\alpha(f)(gx) - \mathfrak{K}_\alpha(f)(x) - (\mathfrak{K}_\alpha(f)(gy) - \mathfrak{K}_\alpha(f)(y)) \\ &= \mathfrak{G}_x(f, h) - \mathfrak{G}_y(f, g). \end{aligned}$$

{E:torus}

□

**Example 3.2.** In this example we show that the boundedness of  $\mathfrak{G}_x$  depends on the choice of a point  $x \in X$ . Let  $X = \mathbf{R}/\mathbf{Z} \times \mathbf{R} \cup \{\infty\}$  be the two-dimensional torus. Let  $\alpha$  be a singular one-cocycle defined by

$$\int_\gamma \alpha := \tilde{\gamma}(1) - \tilde{\gamma}(0),$$

where  $\tilde{\gamma}: [0, 1] \rightarrow \mathbf{R}$  is a lift of the composition of  $\gamma$  followed by the projection onto  $\mathbf{R}/\mathbf{Z}$ . Let  $\alpha$  be the class of  $\alpha$ .

Let  $g \in \text{Homeo}(X, \alpha)$  be a homeomorphism defined by

$$g(t, x) := (t + |x + 1| - |x|, x + 1).$$

Then  $\mathfrak{K}_\alpha(g^n)(t, x) = |x + n| - |x|$  and it follows that

$$\begin{aligned} \mathfrak{G}_{(0,0)}(g^m, g^n) &= \mathfrak{K}_\alpha(g^m)(g^n(0, 0)) - \mathfrak{K}_\alpha(g^m)(0, 0) \\ &= |m + n| - |n| - |m|. \end{aligned}$$

This shows that  $\mathfrak{G}_{(0,0)}$  is unbounded (in fact, the cocycle  $\mathfrak{G}_{(t,x)}$  is unbounded whenever  $x$  is finite). On the other hand,  $g$  acts trivially on the circle  $\mathbf{R}/\mathbf{Z} \times \{\infty\}$  and hence  $\mathfrak{G}_{(t,\infty)} = 0$ . ◇

$\{\mathbf{R} : \text{set}\}$

*Remark 3.3.* Let  $\psi: G \rightarrow \text{Homeo}(X, \mathfrak{a})$  be an action. The set  $\Sigma_\psi$  consisting of points  $x$  for which the cocycle  $\psi^* \mathfrak{G}_x$  is a bounded is an invariant of the action. If  $X$  is compact then it depends on  $\mathfrak{a}$  but not on a continuous representative  $\alpha$ . The above example shows that it can be proper. In Section 3.E we provide an example of an action for which  $\Sigma_\psi = X$ .

Let  $X$  be a compact space. Let us define a pseudo-norm of an element  $g \in \text{Homeo}(X, \mathfrak{a})$  by

$$\|g\|_\alpha := \sup_{x, y \in X} |\mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x)|.$$

This means that  $\|\cdot\|_\alpha$  is symmetric and satisfies the triangle inequality. The finiteness of  $\|g\|_\alpha$  is a consequence of the compactness of  $X$  according to Lemma 2.4. It follows that if  $\Gamma \subset \text{Homeo}(X, \mathfrak{a})$  is a subgroup generated by a finite set  $S$  then

$$C \cdot |g| \geq \|g\|_\alpha,$$

where  $C := \max\{\|s\|_\alpha | s \in S\}$  and  $|g|$  denotes the word norm of  $g \in \Gamma$ . This is just a special case of the standard and straightforward to prove fact that any pseudo-norm on a group is Lipschitz with respect to the word norm.

The next theorem is the main result of this section. Recall that, according to Lemma 2.3 we have that

$$\mathfrak{G}_{x, \alpha}(g, h) = \mathfrak{K}_\alpha(g)(hx) - \mathfrak{K}_\alpha(x).$$

Moreover, it follows from Proposition 3.1 that if  $\mathfrak{G}_x$  and  $\mathfrak{G}_y$  are bounded on the cyclic group  $\langle g \rangle$  generated by a homeomorphism  $g$  then the map  $q: \langle g \rangle \rightarrow \mathbf{R}$  defined by

$$q(g^n) = \mathfrak{K}_\alpha(g^n)(y) - \mathfrak{K}_\alpha(g^n)(x)$$

is a quasimorphism.

$\{\mathbf{T} : \mathbf{q} - \mathbf{m}\}$

**Theorem 3.4.** *Let  $\mathfrak{a} \in H^1(X; \mathbf{R})$  and let  $g \in \text{Homeo}(X, \mathfrak{a})$  and assume that  $X$  is compact. Suppose that for some points  $x, y \in X$  the cocycles  $\mathfrak{G}_x$  and  $\mathfrak{G}_y$  are bounded on the cyclic subgroup  $\langle g \rangle \subset \text{Homeo}(X, \mathfrak{a})$  generated by  $g$ . If the above quasimorphism  $q: \langle g \rangle \rightarrow \mathbf{R}$  is unbounded then  $g$  is undistorted in  $\text{Homeo}(X, \mathfrak{a})$ .*

*Remark 3.5.* It is often the case that to prove that an element  $g$  is undistorted in a group  $G$  one constructs a homogeneous quasimorphism  $q: G \rightarrow \mathbf{R}$  such that  $q(g) \neq 0$ . Constructing such a quasimorphism is in general very difficult. The advantage of the above theorem is that we only need to check that a naturally defined quasimorphism on a cyclic group is unbounded.

*Proof of Theorem 3.4.* Let  $\Gamma' \subseteq \text{Homeo}(X, \alpha)$  be a group containing  $g$  and generated by a finite set  $S' \subseteq \Gamma'$ . Consider the subgroup  $\Gamma \subseteq \text{Homeo}(X, \alpha)$  generated by  $S'$  and  $h$ . It is finitely generated by the set  $S := S' \cup \{h\}$ .

Let  $\hat{q}: \langle g \rangle \rightarrow \mathbf{R}$  be the homogenisation of the quasi-morphism  $q$ . The following calculation of the translation length of  $g$  shows that  $g$  is undistorted in  $\Gamma$  and hence also in  $\Gamma' \subset \Gamma$ .

$$\begin{aligned} C \cdot \tau(g) &= \lim_{n \rightarrow \infty} \frac{C \cdot |g^n|}{n} \\ &\geq \lim_{n \rightarrow \infty} \frac{\|g^n\|_\alpha}{n} \\ &\geq \lim_{n \rightarrow \infty} \frac{|q(g^n)|}{n} \\ &\geq \lim_{n \rightarrow \infty} \frac{n|\hat{q}(g)| - D}{n} \\ &= |\hat{q}(g)| > 0. \end{aligned}$$

Since  $\Gamma'$  is an arbitrary finitely generated subgroup of  $\text{Homeo}(X, \alpha)$ , the element  $g$  is undistorted in  $\text{Homeo}(X, \alpha)$ .  $\square$

proof\_distortion}

**3.B. Proof of Theorem 1.1.** Recall that we need to prove that if  $x$  and  $y$  are fixed points of  $g$  and  $\int_Y g^* \alpha - \alpha \neq 0$  then  $g$  is undistorted in  $\text{Homeo}(X, \alpha)$ .

First observe that the cocycles  $\mathfrak{G}_x$  and  $\mathfrak{G}_y$  vanish identically on the cyclic group  $\langle g \rangle$  because  $x$  and  $y$  are fixed points of  $g$ . By Proposition 3.1 the defect of  $q$  is zero (since it is bounded by  $\|\mathfrak{G}_x\| + \|\mathfrak{G}_y\| = 0$ ) and we obtain that  $q: \langle g \rangle \rightarrow \mathbf{R}$  is a homomorphism of groups. Furthermore

$$\begin{aligned} q(g) &= \mathfrak{K}_\alpha(g)(y) - \mathfrak{K}_\alpha(g)(x) \\ &= \int_Y g^* \alpha - \alpha \neq 0 \end{aligned}$$

according to the hypothesis. Therefore  $q$  is unbounded and the statement follows from Theorem 3.4.  $\square$

{SS:bounded}

**3.C. Bounded cocycles.** In what follows we are interested in bounded cohomology of a group with the integer coefficients; see Gromov [?] and Monod [?] for a background on bounded cohomology.

{E:boundedZ}

**Example 3.6.** (Ghys [?, Section 6.3]) The second bounded cohomology  $H_b^2(B\mathbf{Z}; \mathbf{Z})$  of the integers with integer coefficients is isomorphic to  $\mathbf{R}/\mathbf{Z}$ . To see this let  $\mathfrak{c}: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$  be a bounded two-cocycle. As an ordinary cocycle it is a coboundary since  $H^2(B\mathbf{Z}; \mathbf{Z}) = 0$ . If  $\mathfrak{c} = \delta \mathfrak{b}$  then, since  $\mathfrak{c}$  is bounded, the cochain  $\mathfrak{b}$  is a quasimorphism. The

$$\int_x^{hx} g^* \alpha - \alpha$$

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homogenisation of  $\mathfrak{b}$  (which is a real cochain in general) is given by  $\widehat{\mathfrak{b}}(n) = rn$  for some real number  $r \in \mathbf{R}$ . The required isomorphism

$$H_{\mathfrak{b}}^2(\mathbf{BZ}; \mathbf{Z}) \rightarrow \mathbf{R}/\mathbf{Z}$$

is defined by  $[c] \mapsto r + \mathbf{Z}$ .

◇

Let  $g \in \text{Homeo}(X, \mathfrak{a})$  and let  $x \in X$  be a point for which the cocycle  $\mathfrak{G}_x$  is bounded on the cyclic group generated by  $g$ . The cohomology class  $r_x(g) + \mathbf{Z} \in H^2(\langle g \rangle; \mathbf{Z})$  represented by the pullback of  $\mathfrak{G}_x$  is called the **local rotation number** of  $g$  at the point  $x \in X$ .

Let us explain the geometry of the local rotation number. Take a path  $\eta_{x,1}: [0, 1] \rightarrow X$  from  $x$  to  $gx$  and let  $\eta_{x,n}$  be the concatenation of paths  $g^k(\eta_{x,1})$  for  $k$  ranging from 0 to  $n-1$ . Define a map  $\mathfrak{b}_x: \langle g \rangle \rightarrow \mathbf{R}$  by

$$\mathfrak{b}_x(g^n) := - \int_{\eta_{x,n}} \alpha.$$

Observe that  $\delta \mathfrak{b}_x = \mathfrak{G}_x$  on the cyclic group  $\langle g \rangle$ . Since  $\mathfrak{G}_x$  is bounded on  $\langle g \rangle$  we get that  $\mathfrak{b}_x$  is a quasi-morphism and that its homogenisation satisfies  $\widehat{\mathfrak{b}}_x(g^n) = r_x(g)n$  for a suitable representative of the local rotation number of  $g$  at  $x$ . This shows that there exists a constant  $C_x > 0$  such that

$$|\mathfrak{b}_x(g^n) - r_x(g)n| \leq C_x$$

for all  $n \in \mathbf{Z}$ . We thus obtain that

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{b}_x(g^n)}{n} = r_x(g)$$

and hence the above limit represents the local rotation number of  $g$  at  $x$ . Notice that, since  $\alpha$  has integral periods, the dependence of  $\mathfrak{b}_x(g^n)$  on the choice of the path  $\eta_{x,1}$  is up to an integer constant only. This implies that the above computation of the local rotation number does not depend on the choice of a path  $\eta_{x,1}$ .

*Remark 3.7.* If  $X = \mathbf{S}^1$  then the cocycle  $\mathfrak{G}$  corresponding to the length form is equal to the Euler cocycle. Consequently, the local rotation number defined above equals the classical topological rotation number of a homeomorphism of the circle [?, Section 6.3].

{SS:proof\_rotati

**3.D. Proof of Theorem 1.3.** In order to apply Theorem 3.4 we need to prove that the quasimorphism  $q: \langle g \rangle \rightarrow \mathbf{R}$  from Proposition 3.1 is unbounded.

Let  $\gamma, \eta_{x,n}, \eta_{y,n}: [0, 1] \rightarrow X$  be paths from  $x$  to  $y$ ,  $x$  to  $g^n x$  and  $y$  to  $g^n y$  respectively and  $n \in \mathbf{Z}$ . As above assume that  $\eta_{x,n}$  is a concatenation of the paths  $g^k(\eta_{x,1})$  for  $k$  ranging from 0 to  $n-1$  and similarly for  $\eta_{y,n}$ .

Let  $\mathfrak{b}_x, \mathfrak{b}_y: G \rightarrow \mathbf{R}$  be defined as above and let  $\square_n$  be a concatenation of  $-\gamma$ ,  $\eta_{x,n}$ ,  $g^n\gamma$  and  $-\eta_{y,n}$ . We get the following computation.

$$\begin{aligned} q(g^n) &= \int_Y (g^n)^* \alpha - \alpha \\ &= \int_{\square_n} \alpha - \int_{\eta_{x,n}} \alpha + \int_{\eta_{y,n}} \alpha \\ &= n \int_{\square_1} \alpha + \mathfrak{b}_x(g^n) - \mathfrak{b}_y(g^n) \\ &= n \left( \int_{\square_1} \alpha + (r_x(g) - r_y(g)) \right) + O(1). \end{aligned}$$

Since  $\alpha$  has integral periods and the difference  $r_x(g) - r_y(g) \notin \mathbf{Z}$  by the hypothesis, we get that the quasi-morphism  $q$  is unbounded.  $\square$

{SS:top\_rot}

{C:distorted}

### 3.E. Some consequences of Theorem 1.3.

**Corollary 3.8.** *If  $g \in \text{Homeo}(X; \mathfrak{a})$  is distorted then the local rotation number is constant at all points  $x$  for which  $\mathfrak{G}_x$  is bounded.*  $\square$

{R:gradient}

**Example 3.9.** If  $X$  is a closed oriented surface of a positive genus then  $g \in \text{Homeo}(X)_0$  has a fixed point and hence if it is distorted then the local rotation number of  $g$  has to vanish.  $\diamond$

**Example 3.10.** If  $g$  is a time-one map of a gradient flow then  $\mathfrak{G}_x$  is bounded at every  $x$  and its local rotation number is equal to zero. We do not know whether such elements are distorted or not.  $\diamond$

Let  $G \subset \text{Homeo}(X)$  be a group of homeomorphisms acting trivially on  $H^1(X; \mathbf{R})$ . Let  $g$  be a homeomorphism distorted in  $G$ . Let  $\ell_1, \ell_2 \subset X$  be oriented simple closed curves preserved by  $g$ . We also assume that the classes  $[\ell_i]$  are nonzero in  $H^1(X; \mathbf{R})$ . Let  $\rho_1$  and  $\rho_2$  be the topological rotation numbers associated with the action of  $g$  on  $\ell_1$  and  $\ell_2$  respectively.

{P:top\_rot}

**Proposition 3.11.** *With the above notation and assumptions the following statements are true.*

- (1) *There exist nonzero integers  $k_1, k_2 \in \mathbf{Z}$  such that  $k_1 \rho_1 = k_2 \rho_2$  in  $\mathbf{Q}/\mathbf{Z}$ .*
- (2) *If the classes  $[\ell_1]$  and  $[\ell_2]$  are linearly independent in  $H^1(X; \mathbf{R})$  then both  $\rho_1$  and  $\rho_2$  are rational.*

*Proof.* Let  $\alpha$  be an integral singular one-cocycle and let  $x_i \in \ell_i$ . We get that the topological and the local rotation numbers are related as follows

$$\rho_i \cdot \int_{\ell_i} \alpha = r_{x_i}(g).$$

$$\int_x^{hx} g^* \alpha - \alpha$$

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Since  $g$  is distorted, it follows from Theorem 1.3 that the local rotation numbers  $r_{x_i}(g)$  are equal. Choosing  $\alpha$  such that  $\int_{\ell_i} \alpha \neq 0$  proves the first statement.

To prove the second assertion, we choose  $\alpha$  such that  $\int_{\ell_1} \alpha = 0 \neq \int_{\ell_2} \alpha$ . It follows that  $\rho_2 \cdot \int_{\ell_2} \alpha = 0$  and, since  $\int_{\ell_2} \alpha$  is an integer, it implies that  $\rho_2 \in \mathbf{Q}/\mathbf{Z}$ . The rationality of  $\rho_1$  is proven similarly.  $\square$

#### 4. VANISHING PROPERTIES OF THE COCYCLE $\mathfrak{G}$

{S:vanish}

In this section we prove theorems about the vanishing of the cohomology class represented by the cocycle  $\mathfrak{G}$ .

{SS:proof\_subset}

**4.A. Proof of Theorem 1.5.** (Cf. proof of Theorem 1.3 in [?].) Recall that  $i_L: L \rightarrow M$  is the inclusion of a path-connected subset such that  $i_L^* \mathfrak{a} = 0$ . Therefore, for any  $\alpha$  in the cohomology class  $\mathfrak{a}$  there exists a map  $F: L \rightarrow \mathbf{A}$  such that  $\int_\gamma \alpha = F(\gamma(1)) - F(\gamma(0))$  provided  $\gamma$  is a path contained in  $L$ .

Let us choose a reference point  $x$  in  $L$ . Since the homeomorphism  $h \in G$  preserves  $L$  and  $L$  is path-connected there exists a path in  $L$  from  $x$  to  $hx$ . Therefore

$$\mathfrak{G}(g, h) = F(ghx) - F(gx) - (F(hx) - F(x)).$$

If we define  $\mathfrak{b}(g) := F(x) - F(gx)$  then we get that

$$\mathfrak{G}(g, h) = \mathfrak{b}(g) - \mathfrak{b}(gh) - \mathfrak{b}(h) = \delta \mathfrak{b}(g, h),$$

and hence the cocycle  $\mathfrak{G}$  is cohomologically trivial.  $\square$

**Example 4.1.** The class  $\mathfrak{G}$ , which is equal to the Euler class of  $\text{Homeo}(\mathbf{S}^1)_0$ , is nontrivial on  $\text{PSL}(2, \mathbf{R})$  and restricts to a nontrivial class on any cocompact lattice  $\Gamma \subset \text{PSL}(2, \mathbf{R})$  [?, Section 6.2].

On the other hand, every orbit of  $\Gamma$  is countable thus simply-connected. This shows that the assumption on the connectivity of an invariant subset is essential in Theorem 1.5.  $\diamond$

{SS:proof\_measur}

**4.B. Proof of Theorem 1.6.** Recall that we need to prove that if an action of a group  $G$  preserves a Borel probability measure then the corresponding class  $\psi^*[\mathfrak{G}]$  is trivial.

Let us choose, by Lemma 2.4, a representative  $\alpha$  of  $\mathfrak{a} \in H^1(X, \mathbf{R})$  such that  $\mathfrak{K}_\alpha(h)$  is continuous for each homeomorphism  $h$  of  $X$ . Let  $\mu$  be a Borel probability measure. We define the lift  $\tilde{\mathfrak{K}}_\alpha(h)$  by the following normalisation condition,

$$\int_X \tilde{\mathfrak{K}}_\alpha(h) \mu = 0.$$



This can be done because  $\mathfrak{K}_\alpha(h)$ , being continuous, is integrable.

Since  $G$  preserves a measure  $\mu$  we see that

$$\tilde{\mathfrak{K}}_\alpha(g) \circ h - \tilde{\mathfrak{K}}_\alpha(gh) + \tilde{\mathfrak{K}}_\alpha(h) = 0.$$

Indeed, the left hand side is a constant and integrating it with respect to  $\mu$  we get that this constant is zero. Thus  $\tilde{\mathfrak{K}}_\alpha$  is a one-cocycle with values in  $C^0(X; \mathbf{R})$  lifting the cocycle  $\mathfrak{K}_\alpha$ .

Finally, by Lemma 2.3 we have that

$$\mathfrak{G}(g, h) = -\tilde{\mathfrak{K}}_\alpha(g)(x) + \tilde{\mathfrak{K}}_\alpha(g)(hx) = -\tilde{\mathfrak{K}}_\alpha(g)(x) + \tilde{\mathfrak{K}}_\alpha(gh)(x) - \tilde{\mathfrak{K}}_\alpha(h)(x).$$

This implies that  $\mathfrak{G} = \delta \mathfrak{b}$  for a cochain defined by  $\mathfrak{b}(g) := -\tilde{\mathfrak{K}}_\alpha(g)(x)$ .  $\square$

## 5. COHOMOLOGY CLASS DEFINED BY $\mathfrak{G}$

{S:class}

Recall that we have defined the cocycle  $\mathfrak{G}$  by an explicit formula in Section 2.A and also the cocycle  $\mathfrak{K}_\alpha$  in Section 2.C. The purpose of this section is to give several characterisations of the cohomology class represented by  $\mathfrak{G}$ . This will be used in the next section to prove the nonvanishing results for the cohomology class  $[\mathfrak{G}]$ .

Throughout this section let  $G := \text{Homeo}(X, \mathfrak{a})$  denote the group of homeomorphisms of  $X$  preserving the cohomology class  $\mathfrak{a} \in H^1(X; \mathbf{A})$ .

**5.A. The extension class.** Let  $\mathbf{A} \rightarrow X_\mathfrak{a} \xrightarrow{p} X$  be the covering associated with the cohomology class  $\mathfrak{a} \in H^1(X; \mathbf{A})$ . Let  $G_\mathfrak{a}$  be the group of homeomorphisms of  $X_\mathfrak{a}$  commuting with the deck transformations and projecting onto  $G$ . There is a central extension

$$0 \rightarrow \mathbf{A} \rightarrow G_\mathfrak{a} \rightarrow G \rightarrow 0.$$

Let  $\mathfrak{E}_\mathfrak{a} \in H^2(BG^\delta, \mathbf{A})$  be the corresponding extension class.

**5.B. Transgression.** Consider the following universal fibration

$$X \rightarrow X_{G^\delta} := X \times_{G^\delta} EG^\delta \rightarrow BG^\delta$$

associated with the natural action of  $G^\delta$  on  $X$ . Then the differential

$$d_2: E_2^{0,1} = H^1(X, \mathbf{A})^G \rightarrow H^2(BG^\delta, \mathbf{A}) = E_2^{2,0}$$

in the Leray-Serre spectral sequence defines a cohomology class  $d_2[\mathfrak{a}] \in H^2(BG^\delta; \mathbf{A})$ .

{T:def}

**Theorem 5.1.** *The following equalities hold in  $H^2(BG^\delta, \mathbf{A})$*

$$\delta[\mathfrak{K}_\alpha] = [\mathfrak{G}] = \mathfrak{E}_\mathfrak{a} = d_2\mathfrak{a}.$$

**5.C. Proof of the first equality:**  $\delta[\mathfrak{K}_\alpha] = [\mathfrak{G}]$ . Consider the following extension of G-representations

$$0 \rightarrow \mathbf{A} \rightarrow C^0(X; \mathbf{A}) \rightarrow C^0(X; \mathbf{A})/\mathbf{A} \rightarrow 0.$$

It induces the connecting homomorphism

$$\delta: H^1(BG^\delta; C^0(X; \mathbf{A})/\mathbf{A}) \rightarrow H^2(BG^\delta; \mathbf{A})$$

and hence we obtain the class  $\delta[\mathfrak{K}_\alpha] \in H^2(BG^\delta; \mathbf{A})$ .

Take a section  $C^0(X; \mathbf{A})/\mathbf{A} \rightarrow C^0(X; \mathbf{A})$  that chooses a function vanishing at the basepoint  $x$ . Denote by  $\tilde{\mathfrak{K}}_\alpha$  the lift of the cocycle to  $C^0(X; \mathbf{A})$  using this section. The function

$$(g, h) \mapsto \tilde{\mathfrak{K}}_\alpha(g) \circ h - \tilde{\mathfrak{K}}_\alpha(gh) + \tilde{\mathfrak{K}}_\alpha(h)$$

is constant and equal to  $(\partial \tilde{\mathfrak{K}}_\alpha)(g, h)$ , the value of a cocycle representing the class  $\partial[\mathfrak{K}_\alpha]$ . Evaluating it at the basepoint we get that

$$\partial \tilde{\mathfrak{K}}_\alpha(g, h) = \tilde{\mathfrak{K}}_\alpha(g)(hx) = \int_\gamma g^* \alpha - \alpha = \mathfrak{G}_x(g, h),$$

where  $\gamma$  is any path between  $x$  and  $hx$ .  $\square$

**5.D. Proof of the second equality:**  $[\mathfrak{G}] = \mathfrak{E}_a$ . Let  $\mathbf{a}: X_a \rightarrow \mathbf{A}$  be an equivariant map representing the cohomology class  $\mathfrak{a}$  as in Lemma 2.2. Fix a reference point  $\tilde{x} \in p^{-1}(x) \subset X_a$  with  $\mathbf{a}(\tilde{x}) = 0$ . Consider the extension  $\mathbf{A} \rightarrow G_a \rightarrow G$  and let  $\tilde{\cdot}: G \rightarrow G_a$  be a section defined by

$$(5.2) \quad \{\text{Eq:section}\} \quad \mathbf{a}(\tilde{f}\tilde{x}) = 0.$$

Since  $\tilde{f}$  commutes with the action of  $\mathbf{A}$ , it follows that if  $\tilde{y}_1$  and  $\tilde{y}_2$  are two points in the same fibre of  $X_a \rightarrow X$  we have

$$(5.3) \quad \{\text{Eq:fibre}\} \quad \mathbf{a}(\tilde{f}\tilde{y}_1) - \mathbf{a}(\tilde{y}_1) = \mathbf{a}(\tilde{f}\tilde{y}_2) - \mathbf{a}(\tilde{y}_2).$$

Let  $\gamma: [0, 1] \rightarrow X$  be a curve from  $x$  to  $gx$ . Let  $\tilde{\gamma}: [0, 1] \rightarrow X_a$  be a lift with  $\tilde{\gamma}(0) = \tilde{x}$ . Let  $\tilde{y}_1 = \tilde{g}\tilde{x}$  and  $\tilde{y}_2 = \tilde{\gamma}(1)$ . Then we obtain the following equalities

$$(5.4) \quad \{\text{Eq:3.4}\} \quad \mathbf{a}(\tilde{f}\tilde{g}\tilde{x}) = \mathbf{a}(\tilde{f}\tilde{g}\tilde{x}) - \mathbf{a}(\tilde{g}\tilde{x}) = \mathbf{a}(\tilde{f}\tilde{\gamma}(1)) - \mathbf{a}(\tilde{\gamma}(1)).$$

where the first equality follows from (5.2) and the second one from (5.3).

Since  $\tilde{f}\tilde{\gamma}$  is a lift of  $f\gamma$  starting at  $\tilde{f}\tilde{\gamma}(0) = \tilde{f}\tilde{x}$ , we have that

$$\mathbf{a}(\tilde{f}\tilde{\gamma}(1)) = \int_{f\gamma} \alpha,$$

where  $\alpha$  and  $\mathbf{a}$  are related as in Lemma 2.2. This together with (5.4) implies that

$$(5.5) \quad \{\text{Eq:3.5}\} \quad \mathbf{a}(\tilde{f}\tilde{g}\tilde{x}) = \int_{f\gamma} \alpha - \int_\gamma \alpha = \mathfrak{G}(f, g).$$

The extension class is represented by the cocycle defined by the following identity [?, Section IV.3]:

$$(5.6) \quad \{\text{Eq: brown}\} \quad \mathfrak{E}(f, g) + \widetilde{f}g = \widetilde{f}\widetilde{g}.$$

In the following calculation the second equality follows from the equivariance of  $\mathbf{a}$  and the others as marked.

$$\begin{aligned} \mathfrak{E}(f, g) &= \mathfrak{E}(f, g) + \mathbf{a}(\widetilde{f}g\widetilde{x}) && \text{by (5.2)} \\ &= \mathbf{a}(\mathfrak{E}(f, g) + \widetilde{f}g\widetilde{x}) \\ &= \mathbf{a}(\widetilde{f}\widetilde{g}\widetilde{x}) && \text{by (5.6)} \\ &= \mathfrak{E}(f, g) && \text{by (5.5)} \end{aligned}$$

**5.E. Proof of the third equality:**  $\mathfrak{E}_a = d_2[a]$ . The Lyndon-Hochschild-Serre spectral sequence associated to the extension  $\mathbf{A} \rightarrow G_a \rightarrow G$  is isomorphic to the Leray-Serre spectral sequence associated with the fibration

$$\mathbf{BA} \rightarrow \mathbf{BG}_a^\delta \rightarrow \mathbf{BG}^\delta.$$

We shall make all the computations in the latter. We have isomorphisms

$$H^0(\mathbf{BG}^\delta; H^1(\mathbf{BA}; \mathbf{A})) = H^1(\mathbf{BA}; \mathbf{A})^G = \text{Hom}(\mathbf{A}; \mathbf{A})^G$$

and, using this identifications, the extension class is defined to be

$$\mathfrak{E}_a := d_2[\text{id}]$$

$\{\text{P: ext}\}$  where  $d_2: H^0(\mathbf{BG}^\delta; H^1(\mathbf{BA}; \mathbf{A})) \rightarrow H^2(\mathbf{BG}^\delta; H^0(\mathbf{BA}; \mathbf{A})) = H^2(\mathbf{BG}^\delta; \mathbf{A})$  is the differential in the spectral sequence.

**Proposition 5.7.** *Let  $X \rightarrow E := \mathbf{EG}^\delta \times_{G^\delta} X \rightarrow \mathbf{BG}^\delta$  be the universal bundle associated with the action of  $G^\delta$  on  $X$ . Then the extension class*

$$\mathfrak{E}_a = d_2(a)$$

where  $d_2: H^1(X; \mathbf{A})^G \rightarrow H^2(\mathbf{BG}^\delta; \mathbf{A})$  is the differential in the associated spectral sequence.

*Proof.* Since there is an isomorphism  $H^1(X; \mathbf{A}) = [X, \mathbf{BA}]$ , the class  $a$  can be represented by a continuous map  $\alpha: X \rightarrow \mathbf{BA} = K(\mathbf{A}, 1)$ . Thus  $a = \alpha^*[\text{id}]$ . Let

$$E_a := \mathbf{EG}_a^\delta \times_{G_a^\delta} X_a$$

and let us consider the following diagram of fibrations.

$$\begin{array}{ccccccc}
 \mathbf{A} & \longrightarrow & X_{\mathfrak{a}} & \longrightarrow & X & \xrightarrow{\alpha} & \mathbf{BA} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{EA} & \longrightarrow & E_{\mathfrak{a}} & \longrightarrow & E & \xrightarrow{\mathfrak{A}} & \mathbf{BG}_{\mathfrak{a}}^{\delta} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{BA} & \longrightarrow & \mathbf{BG}_{\mathfrak{a}}^{\delta} & \longrightarrow & \mathbf{BG}^{\delta} & \equiv & \mathbf{BG}^{\delta}
 \end{array}$$

Since the fibration  $\mathbf{EA} \rightarrow E_{\mathfrak{a}} \rightarrow E$  has a contractible fibre, it admits a section. The map  $\mathfrak{A}: E \rightarrow \mathbf{BG}_{\mathfrak{a}}^{\delta}$  is defined as the composition of this section followed by the projection. In this way the two right-hand side columns form a morphism of bundles. Hence the result follow from the functoriality of the spectral sequence.  $\square$

## 6. NON-TRIVIALITY OF THE COCYCLE $\mathfrak{G}$

$\{\text{S5nprvafish}\}$

**6.A. Proof of Theorem 1.9.** The idea of the proof is that the cohomology class  $\psi^*[\mathfrak{G}]$  is equal to the transgression of  $\mathfrak{a}$  in the universal bundle  $X \rightarrow X_{G^{\delta}} \rightarrow \mathbf{BG}^{\delta}$  according to Theorem 5.1. This implies that it is the image of the corresponding transgression for the universal bundle  $X \rightarrow X_G \rightarrow \mathbf{BG}$  for the action of the connected group  $G$ . The transgression for a connected group is then related with the topology of the corresponding evaluation map.

More precisely, the identity homomorphism  $G^{\delta} \rightarrow G$  induces a continuous map  $\beta: \mathbf{BG}^{\delta} \rightarrow \mathbf{BG}$ . Consider the following morphism of universal bundles.

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \downarrow & & \downarrow \\
 X_{G^{\delta}} & \longrightarrow & X_G \\
 \downarrow & & \downarrow \\
 \mathbf{BG}^{\delta} & \xrightarrow{\beta} & \mathbf{BG}
 \end{array}$$

It follows from Theorem 5.1 that  $\psi^*[\mathfrak{G}] = \beta^*(d_2(\mathfrak{a}))$ .

Let  $\text{ev}: G \rightarrow X$  denote the evaluation at the reference point  $x \in X$  associated with the action. We have yet another morphism of universal

bundles.

$$\begin{array}{ccc}
 G & \xrightarrow{\text{ev}} & X \\
 \downarrow & & \downarrow \\
 EG & \longrightarrow & X_G \\
 \downarrow & & \downarrow \\
 BG & \xlongequal{\quad} & BG
 \end{array}$$

The evaluation map induces the morphism of spectral sequences that on the second page is the map

$$\mathcal{E}^{p,q} : H^p(BG; H^q(X; \mathbf{A})) \rightarrow H^p(BG; H^q(G; \mathbf{A}))$$

defined by  $\text{ev}^*$  on the coefficients level. It follows from the connectivity of  $G$  that  $\mathcal{E}^{p,0}$  is injective. In both spectral sequences we denote the differential on the second page by  $d_2$ . We have the following straightforward equalities.

$$d_2(\text{ev}^*(\alpha)) = d_2(\mathcal{E}^{0,1}(\alpha)) = \mathcal{E}^{2,0}(d_2(\alpha))$$

Since  $EG$  is contractible the corresponding differential

$$d_2 : H^0(BG; H^1(G; \mathbf{A})) \rightarrow H^2(BG; H^0(G; \mathbf{A}))$$

is an isomorphism. This, together with the injectivity of  $\mathcal{E}^{p,0}$ , implies that  $\text{ev}^*(\alpha) \neq 0$  if and only if  $d_2(\alpha) \neq 0$  which finishes the proof of Theorem 1.9.  $\square$

**6.B. Remark on Corollary 1.10.** The result of McDuff in [?] states that the comparison map  $\beta : B\text{Homeo}(X)^\delta \rightarrow B\text{Homeo}(X)$  is a homology equivalence for the full group of homeomorphisms. It is however known that that the homotopy fibre of  $\beta$  is determined by the topological and algebraic structure of a neighbourhood of the identity. This implies that  $\beta$  is a homology equivalence for any group of homeomorphisms containing the connected component of the identity as a subgroup. In particular it holds for  $\text{Homeo}(X, \mathfrak{a})$ .

{SS:perfect}

**6.C. Proof of Corollary 1.11.** Let  $\tilde{G} \rightarrow G$  be the universal cover. It follows from the countability of the fundamental group  $\pi_1(G)$  that  $\tilde{G}$  is perfect. Indeed, let  $\text{Ab} : \tilde{G} \rightarrow H := \tilde{G} / [\tilde{G}, \tilde{G}]$  be the abelianisation. It induces a surjective map  $G \rightarrow H / \text{Ab}(\pi_1(G))$  which is trivial because  $G$  is perfect. This implies that  $H = \text{Ab}(\pi_1(G))$  and since  $H$  is path connected it must be trivial.

Now we follow the proof of Lemma 6 in Milnor [?]. Let  $\mathcal{F}$  denote the homotopy fibre of the comparison map  $\beta : BG^\delta \rightarrow BG$ . Since it depends only on the local structure of the group it is also the homotopy fibre of the corresponding comparison map for the universal cover. It follows from the perfectness of  $\tilde{G}$  and the spectral sequence for the fibration

$\mathcal{F} \rightarrow B\tilde{G}^\delta \rightarrow B\tilde{G}$  that  $H^1(\mathcal{F}; \mathbf{Z}) = 0$ . This implies that  $H^1(\mathcal{F}; \mathbf{A}) = 0$  by the universal coefficients theorem. It then follows from the spectral sequence for the fibration  $\mathcal{F} \rightarrow B\tilde{G}^\delta \rightarrow BG$  that the homomorphism  $\beta^*: H^2(BG; \mathbf{A}) \rightarrow H^2(B\tilde{G}^\delta; \mathbf{A})$  is injective.  $\square$

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